Flux Limited Diffusion in Hydrodynamics

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Abstract

The radiation transport equation, correct to order v/c, is used to to derive a flux limiter valid in high velocity hydrodynamic flows. In the limit $v = 0$ the widely used result of Levermore and Pomraning (**LP**) is recovered. In the case of homologous flow ($v \propto R$) the **LP** result remains valid, but in the presence of shock waves and other non–homologous flows differences may be large. We calculate the Eddington factors, which are significantly altered, and discuss the importance of having them possess the correct form in the presence of a shock wave, e.g. on the neutrino transport during the formation of a supernova.

1 Introduction

Radiative transfer problems occur in a wide range of physical and astrophysical situations. Types of radiation considered have included photons, neutrons, and more recently neutrinos within the framework of simulations of supernova explosions and neutron star birth. The equation of radiative transport is notoriously difficult to solve, particularly when one has to solve it many times in the course of calculating hydrodynamic flow. Only with the recent increase in computer power available for supernova simulations, could one even contemplate doing one dimensional hydrodynamics with a realistic equation of state and solving for the angular dependence of the specific intensity of radiation.

For computational expediency the usual approach has been to close the moment equations by the means of Eddington factors which relate the flux and the radiation pressure to the energy density. This method goes under the general heading of flux limited diffusion, since the moment equations are closed by assuming a Fick's law diffusion relation. Thus, one is left with the problem of specifying a diffusion coefficient and an Eddington factor relating the radiation pressure to the energy density. A variety of flux limiters have been used in practice (*cf.* Pomraning 1982; Bruenn, Buchler, and Yueh 1978; Bowers and Wilson 1982; Bruenn 1985).

One of the most successful and often used is the flux limiter derived by Levermore and Pomraning (1981) (**LP**). In this case the flux limiter is derived from the transport equation itself. However, **LP** begin with the equations for a static medium and hence one is not even trying to approximate the correct equation once material velocities are present. Clearly, the relativistic corrections corresponding to the Doppler shift and advection or relativistic aberration are neglected. While it is common practice (*cf.* Baron *et al.* 1989) to use this flux limiter together with the fully general relativistic equations (correct to all orders in v/c) this is clearly a mistake, because it is well known that energy and momentum conservation is lost when the v/c terms are neglected in the transport equation (Castor 1972; Mihalas and Mihalas 1984). While there has been much discussion in the literature about various forms of the flux limiter in the static case, we will show that the changes due to material velocities may be much more important than the variations in different static prescriptions.

Generalizing the method of **LP**, we derive a flux limiter beginning with the correct form of the transfer equation in the co-moving frame. In §2 we derive the two Eddington factors for the case of grey opacities. The non-grey, or frequency dependent, case is derived in an appendix, and the results are seen to be essentially the same as the grey case with some redefinition of parameters. In §3 we discuss the solution of the transcendental equations specifying the Eddington factors obtained in the previous section, and give analytical forms for regions in which various approximations are valid. In §4 we compare our results to those of **LP**. In §5 we present an example of our results within a hydrodynamic calculation of a supernova explosion. In §6 we discuss the uniqueness of our choice of solution of the transport equation, and in a final section we present our conclusions.

While we will concentrate on neutrinos as the radiation considered when we come to giving examples, our treatment is perfectly general, and can be used whenever radiative transport occurs within a hydrodynamic context, such the case of photons in pulsating stars or in the neighborhood of accretion disks.

We will restrict our attention to the spherically symmetric case, in contrast to the **LP** result which can be applied to less restricted geometries. We suspect that our treatment can be generalized to other geometries but the integrals obtained in the generalized case may no longer be possible to perform analytically. The present work presents the logical departure point for such a study.

2 Transfer Equation

We begin with the equation of radiative transfer correct to order v/c (Castor 1972; Mihalas and Mihalas 1984):

$$\begin{aligned}\frac{dI_\nu}{dt} + \mu\frac{\partial I_\nu}{\partial R} &+ (1-\mu^2)\left(\mu\left(\frac{v}{R} - \frac{\partial v}{\partial R}\right) + \frac{1}{R}\right)\frac{\partial I_\nu}{\partial \mu} \\ &+ \left(\mu^2\left(\frac{v}{R} - \frac{\partial v}{\partial R}\right) - \frac{v}{R}\right)\left(\nu\frac{\partial I_\nu}{\partial \nu} - 3I_\nu\right) \\ &= \kappa_A(B_\nu - I_\nu) + \kappa_S(\varepsilon_\nu - I_\nu)\end{aligned} \tag{1}$$

where I_ν is the specific intensity at frequency ν and polar angle $\cos\theta = \mu$. We have assumed the intensity is independent of azimuthal angle. Then the zeroth moment with respect to μ gives

$$\varepsilon_\nu = \frac{1}{2}\int_{-1}^{+1} d\mu\, I_\nu\ , \tag{2}$$

the energy density at frequency ν. The radius is denoted by R, the matter velocity is given by $v = dR/dt$, $d/dt = \partial/\partial t + v\partial/\partial R$ is the Lagrangean time derivative, and we have set $c = 1$. κ_A is the absorption opacity, and κ_S is the scattering opacity at frequency ν. The equilibrium energy density is B_ν and in writing Equation (1) we have assumed the Kirchhoff–Planck relation $\eta_\nu = \kappa_A B_\nu$, where η_ν is the emission coefficient at frequency ν. In obtaining Equation (1) we have used the equation of continuity

$$\frac{\partial v}{\partial R} + \frac{2v}{R} = -\frac{1}{\rho}\,\frac{d\rho}{dt} \tag{3}$$

where ρ is the material density.

We have dropped terms in Equation (1) proportional to the acceleration $a = dv/dt$ since formally these terms are of order $(v/c)^2$ and as long as the velocity is not evolving on a radiation flow time scale, as is true for most astrophysical situations, they can be dropped (Mihalas and Mihalas 1984).

We now restrict ourselves to grey opacities and integrate over frequency; in practice the opacity would correspond to the Rosseland mean opacity. The non-grey case is discussed in the appendix and will be seen to give merely a redefinition of parameters. Integrating from $\nu = 0$ to $\nu = \infty$, Equation (1) becomes

$$\begin{aligned}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial R} + (1-\mu^2)\frac{1}{R}\frac{\partial I}{\partial \mu} &+ \left(\frac{v}{R} - \frac{\partial v}{\partial R}\right)\left((1-\mu^2)\mu\frac{\partial I}{\partial \mu} - 4\mu^2 I\right) + 4\frac{v}{R}I \\ &= \kappa_A(B - I) + \kappa_S(\varepsilon - I)\end{aligned} \tag{4}$$

where $I,\ B,\ \varepsilon$ are frequency integrated quantities. Angular integration over μ of I gives the moments

$$\begin{aligned}
\varepsilon &= \frac{1}{2}\int_{-1}^{+1} d\mu\, I & (5)\\
f\ \varepsilon &= \frac{1}{2}\int_{-1}^{+1} d\mu\, \mu I = \mathcal{F} & (6)\\
\chi\ \varepsilon &= \frac{1}{2}\int_{-1}^{+1} d\mu\, \mu^2 I = P\ , & (7)
\end{aligned}$$

which define the Eddington factors f and χ which relate $\mathcal{F}$, the energy flux, and P, the radiation pressure, to ε, the energy density.

We will now integrate Equation (4) over angle to obtain the energy conservation equation (the first thermodynamic law). Using

$$\begin{aligned}
\frac{1}{2}\int_{-1}^{+1} d\mu\, \mu(1-\mu^2)\frac{\partial I}{\partial \mu} &= (3\chi-1)\varepsilon \\
\frac{1}{2}\int_{-1}^{+1} d\mu\, (1-\mu^2)\frac{\partial I}{\partial \mu} &= 2f\varepsilon & (8)
\end{aligned}$$

we obtain

$$\begin{aligned}
\frac{\partial \varepsilon}{\partial t} + \frac{\partial f\varepsilon}{\partial R} + 2\frac{f\varepsilon}{R} - \left(\frac{v}{R} - \frac{\partial v}{\partial R}\right)(1+\chi)\varepsilon &+ 4\frac{v}{R}\varepsilon \\
&= \kappa_A(B-\varepsilon)\ . & (9)
\end{aligned}$$

In the spirit of **LP** we now introduce the product form for I,

$$I \equiv \varepsilon(R,\, t)\psi(\mu,\, R,\, t) \qquad (10)$$

normalized according to

$$\frac{1}{2}\int_{-1}^{+1} d\mu\, \psi = 1\ . \qquad (11)$$

While there is no obvious reason why the ansatz of Equation (10) should be a good one in the case $v \neq 0$, and perhaps a different ansatz including velocities could be employed, it is the straightforward way to generalize **LP**. Introducing the product form into Equation (4), we obtain

$$\begin{aligned}
\varepsilon\frac{\partial \psi}{\partial t} &+ \psi\frac{\partial \varepsilon}{\partial t} + \mu\varepsilon\frac{\partial \psi}{\partial R} + \mu\psi\frac{\partial \varepsilon}{\partial R} \\
&+ (1-\mu^2)\frac{\varepsilon}{R}\frac{\partial \psi}{\partial \mu} + \left(\frac{v}{R} - \frac{\partial v}{\partial R}\right)\varepsilon\left((1-\mu^2)\mu\frac{\partial \psi}{\mu} - 4\mu^2\psi\right) + 4\frac{v}{R}\varepsilon\psi \\
&= \kappa_A(B-\varepsilon\psi) + \kappa_S(1-\psi)\varepsilon\ . & (12)
\end{aligned}$$

Inserting Equation (9) into Equation (12) and dividing by ε we obtain

$$
\begin{aligned}
& \left\{ \frac{\partial \psi}{\partial t} + \mu \frac{\partial \psi}{\partial \mu} + \frac{1-\mu^2}{R} \frac{\partial \psi}{\partial \mu} + \left(\frac{v}{R} - \frac{\partial v}{\partial R} \right) \mu (1-\mu^2) \frac{\partial \psi}{\partial \mu} \right\} \\
+ \quad \psi \quad & \left\{ \frac{\mu - f}{\varepsilon} \frac{\partial \varepsilon}{\partial R} - \frac{\partial f}{\partial R} - 2\frac{f}{R} + \left(\frac{v}{R} - \frac{\partial v}{\partial R} \right) (1 + \chi - 4\mu^2) + a\kappa \right\} = a\kappa \qquad (13)
\end{aligned}
$$

where a, the effective albedo of **LP**, and κ, the total opacity, are defined by

$$a = \frac{\kappa_A B + \kappa_S \varepsilon}{\kappa \varepsilon} \qquad (14)$$

$$\kappa = \kappa_A + \kappa_S \ . \qquad (15)$$

Up to this point we still have a formal solution to the exact transport equation, Equation (4), there being no new content explicitly introduced by the definition of Equation (10). In the spirit of **LP**, we now require that both terms in squiggly brackets in Equation (13) vanish identically; this is a direct generalization of their derivation to inclusion of the time dependent terms in R and v. This split is of course not unique; see Cernohorsky and van den Horn (1990) for a discussion of "gauge freedom" in flux limiting. We will discuss this question later in §6 and find the present choice to be a good physical one.

If we require the first term in squiggly brackets in Equation (13) to vanish and integrate over angle we obtain the equation

$$\frac{\partial f}{\partial R} + \frac{2f}{R} = -\left(\frac{v}{R} - \frac{\partial v}{\partial R} \right) (3\chi - 1) \ , \qquad (16)$$

which is a generalization of the **LP** result that f be divergence–free. Inserting this into the equation generated by requiring the second squiggly bracket in Equation (13) to vanish gives our basic result:

$$\psi(\mu) = \frac{1}{1 + A(f - \mu) + B(\chi - \mu^2)} \qquad (17)$$

where the dimensionless parameters A and B are given by:

$$A = \frac{1}{\varepsilon a \kappa} \left(-\frac{\partial \varepsilon}{\partial R} \right) \qquad (18)$$

$$B = \frac{4}{c a \kappa} \left(\frac{v}{R} - \frac{\partial v}{\partial R} \right) \qquad (19)$$

where we have explicitly displayed the dimension factor c into the definition of B. In the limit that B vanishes the **LP** flux limiter is recovered. Equation (17) is the central result of this paper.

In the appendix, we solve the non–grey, frequency dependent case, and find that the angular distribution function ψ retains the same form as in Equation (17) with the redefinition

$$B \longrightarrow B(3 - \frac{\partial \ln \varepsilon_\nu}{\partial \ln \nu})/4 \ , \tag{20}$$

and the following analysis still follows with this redefinition in mind. Further analysis is made in the appendix.

The parameters A and B have direct physical meaning. Considering the case $a = 1$ for simplicity, we note that the mean free path $\lambda = 1/\kappa$. Noting that the diffusion flux is given by

$$\mathcal{F}_{\rm diff} = -\frac{\lambda}{3}\frac{\partial \varepsilon}{\partial R} \tag{21}$$

and the free streaming flux is given by $\mathcal{F}_{\rm free} = \varepsilon$, we find

$$A = 3\,\frac{\mathcal{F}{\rm diff}}{\mathcal{F}{\rm free}} \ , \tag{22}$$

which is equivalent to Levermore and Pomraning's parameter **R**. The parameter B can be written as

$$B = \frac{4\lambda}{c}\left(\frac{v}{R} - \frac{\partial v}{\partial R}\right) \ , \tag{23}$$

and can be quite large in the free streaming limit (large λ) or when the factor $(\frac{v}{R} - \frac{\partial v}{\partial R})$ has large magnitude. Note that in the case of homologous flow we have $v \propto R$ so B vanishes (This can also be seen by noting that in homologous flow, each mass element has constant ρR^3.)

In the **LP** case ($B = 0$) the diffusion limit is obtained when $A << 1$ and the free streaming limit when $A >> 1$, and thus causality of the flow is respected. In the case of non–vanishing B causality is also respected as we now show. We have the relations

$$1 = \frac{1}{2}\int_{-1}^{+1} d\mu\, \psi \tag{24}$$

$$f = \frac{1}{2}\int_{-1}^{+1} d\mu\, \mu\psi \tag{25}$$

$$\chi = \frac{1}{2}\int_{-1}^{+1} d\mu\, \mu^2\psi \ . \tag{26}$$

Noting that $|\mu| \leq 1$ and that ψ must be positive definite (see the discussion of Equation (39)), comparison of Equation (24) and Equation (25) yields the causality constraint

$$|f| \leq 1 \ , \tag{27}$$

for all values of A and B. Likewise the following quantity must be positive–definite:

$$\frac{1}{2}\int_{-1}^{+1} d\mu\, (\mu - f)^2\psi \geq 0 \ . \tag{28}$$

Expanding out this becomes

$$\chi - 2f^2 + f^2 \geq 0 \ , \tag{29}$$

which together with the bound on f yields

$$1 \geq \chi \geq f^2 \tag{30}$$

which ensures that the pressure be positive definite. Note, however, that there is no restriction that $\chi \geq 1/3$ as in the case $B = 0$.

The standard flux–limiting treatments, both those based on **LP** and others that are widely used, do not take into account non–vanishing B. As such they are not approximations to the correct transport equation, Equation (1), even in a formal sense, *in any regime*. One particular region in which B does not vanish is in the region of a shock wave. In the case of an accretion shock for instance, we have $\frac{\partial v}{\partial R} >> \frac{v}{R}$ and the corrections to the flux limiter will be large. We will investigate this in detail in §5.

Equation (17) contains two input parameters, A and B and two unknown quantities, f and χ which we must solve for. These two unknown functions must be obtained by solving the pair of non-linear equations, Equations (24) and (25). (The additional moment, Equation (26) generates no additional information as can be verified by direct solution.) The integrals in Equations (24) and (25) can be done analytically, and after some manipulation the following pair of transcendental equations is obtained:

$$4Bw \coth(w) = w^2 - A^2 + 4B^2 \tag{31}$$

$$4BA \coth(A + 2Bf) = w^2 - A^2 - 4B^2 \ , \tag{32}$$

which depend on the parametric variable

$$w^2 = A^2 + 4B(1 + Af + B\chi) \ . \tag{33}$$

Instead of Equation (32) an equivalent form which is generally more convenient is

$$f = \frac{1}{4B}\left(\log\left|\frac{w\coth(w) - 2B + A}{w\coth(w) - 2B - A}\right| - 2A\right) = \frac{1}{4B}\left(\log\left|\frac{w^2 - (A - 2B)^2}{w^2 - (A + 2B)^2}\right| - 2A\right) \tag{34}$$

We note that there is no restriction that $w^2 > 0$ so that w may be imaginary. If we interpret $w = (|w^2|)^{1/2}$ we should adopt

$$w\coth(w) \longrightarrow w\cot(w)$$

in the case $w^2 < 0$, to keep everything real. Solution of the transcendental equations specify f and w, and then χ must be obtained using Equation (33).

In the limit of vanishing B, the **LP** solution can be obtained by expanding Equation (31) and Equation (32) with some care, yielding

$$B = 0: \quad f = \coth(A) - 1/A, \qquad \chi = \coth(A)(\coth(A) - 1/A) \ . \tag{35}$$

3 Eddington Factors

The transcendental equations that supply the Eddington factors f and χ, Equation (31) and Equation (32), are somewhat complicated to solve, given the fact what we may have $w^2 < 0$ and multiple roots exist. Examining Equations (24) and (25) we easily obtain the reflection property

$$f(-A,B) = -f(A,B) \ , \qquad \chi(-A,B) = \chi(A,B) \ , \tag{36}$$

which enables us to restrict our attention to the case $A > 0$, but we still must consider B to have arbitrary sign. Using Equation (33) we can write the angular function of Equation (17) as

$$\psi(\mu) = \frac{4B}{w^2 - (A + 2B\mu)^2} \ . \tag{37}$$

Any physically reasonable angular function must satisfy $\psi \geq 0$ since the radiation intensity must be positive definite. This leads to the restrictions for valid physical solutions,

$$B > 0 \ ; \qquad w^2 \geq (|A| + |2B|)^2 \tag{38}$$

$$B < 0 \ ; \qquad w^2 \leq (|A| + |2B|)^2 \ . \tag{39}$$

Thus for $B > 0$ we must have real solutions for w, but for $B < 0$ there will be cases of both real and imaginary values of w. If we define the quantity

$$w_0^2 \equiv A^2 + 4B(1 - B) \tag{40}$$

and examine Equation (31), noting that for real w the quantity $w\coth(w)$ is always greater than unity, we ascertain that the conditions for the imaginary branch are:

$$B < 0 \ ; \qquad w_0^2 < 0 \ . \tag{41}$$

First we consider the case of $|w^2| << 1$. Then we can expand the factor

$$w\coth(w) \sim 1 + \frac{w^2}{3} - \frac{w^4}{45} + \ldots \tag{42}$$

in Equation (31). After some tedious algebra, we obtain the expansions

$$\begin{aligned} f \ \sim \ & A\left(\frac{1}{3} + \frac{8B}{45} + \frac{76B^2}{945} + \frac{64B^3}{2025} + \frac{1696B^4}{155925} + \frac{696064B^5}{212837625} + \frac{108352B^6}{127702575} + \frac{18102272B^7}{97692469875}\right) \\ - \ & A^3\left(\frac{1}{45} + \frac{16B}{315} + \frac{944B^2}{14175} + \frac{10112B^3}{155925} + \frac{733648B^4}{14189175} + \frac{1044992B^5}{29469825}\right) \\ + \ & A^5\left(\frac{2}{945} + \frac{16b}{1575} + \frac{1292B^2}{51975} + \frac{1072256B^3}{25540515}\right) - A^7\left(\frac{1}{4725} + \frac{32B}{18711}\right) \end{aligned} \tag{43}$$

$$
\begin{aligned}
\chi &\sim \frac{1}{3} \\
&+ \frac{4B}{45} + \frac{16B^2}{945} + \frac{16B^3}{14175} - \frac{64B^4}{93555} - \frac{69248B^5}{212837625} - \frac{512B^6}{8292375} + \frac{211712B^7}{32564156625} + \frac{336729088B^8}{38979295480125} \\
&+ A^2\left(\frac{4}{45} + \frac{76B}{945} + \frac{32B^2}{675} + \frac{3392B^3}{155925} + \frac{348032B^4}{42567525} + \frac{108352B^5}{42567525} + \frac{9051136B^6}{13956067125}\right) \\
&- A^4\left(\frac{4}{315} + \frac{472B}{14175} + \frac{2528B^2}{51975} + \frac{733648B^3}{14189175} + \frac{261248B^4}{5893965}\right) \\
&+ A^6\left(\frac{8}{4725} + \frac{1292B}{155925} + \frac{536128B^2}{25540515}\right) - A^8\frac{4}{18711}
\end{aligned} \tag{44}
$$

which are correct to eighth order in A and B. The expansions are valid when

$$
|w_0^2| << 1 \ . \tag{45}
$$

(These results are also valid as written in the case $A < 0$ as the obey the reflection property, Equation (36).) These asymptotic expressions give the leading order corrections to the opaque diffusion limit, where $f \sim 0$ and $\chi \sim 1/3$ and reduce to the analogous expansions of Equation (35) when $B = 0$.

Noting that $\mathcal{F}$ is defined in the co–moving frame, the flux in the Eulerian, or lab, frame is given by

$$
\mathcal{F}_{\mathrm{Euler}} = \mathcal{F} + v(\varepsilon + \chi\varepsilon) \ , \tag{46}
$$

and using the expansions of Equation (44) we find

$$
\mathcal{F}_{\mathrm{Euler}} \sim -\frac{\lambda}{3}\frac{\partial\varepsilon}{\partial R} + \frac{4}{3}v\varepsilon + \frac{4B}{15}\left(-2\frac{\lambda}{3}\frac{\partial\varepsilon}{\partial R} + \frac{v\varepsilon}{3}\right) \ . \tag{47}
$$

Thus, in the limit $B = 0$, the proper diffusive limit is obtained in the lab frame.

In the case of $|w^2| >> 1$ both real and imaginary solutions may exist. Focusing on the real case (which always obtains when $B > 0$) we rewrite Equation (31) as

$$
(w - 2B)^2 = A^2 + 4Bw(\coth(w) - 1) \tag{48}
$$

We assume $w^2 > 0$ and $w > 0$. In this case we make use of the expansion

$$
\coth(w) \sim 1 + 2e^{-2w} \tag{49}
$$

which gives

$$
w \sim (2B \pm A)(1 + \frac{4B}{A}e^{-4B\mp 2A}) \tag{50}
$$

where we are taking as a small quantity the factor e^{-2w}. Noting the constraints of Equation (39) we see we have to choose the upper sign, or

$$
w^2 \sim (A + 2B)^2\,(1 + \frac{8B}{A}e^{-2A-4B}) \ . \tag{51}
$$

Inserting this into Equation (34) we find

$$f \sim 1 - \frac{1}{2B} \log \left| 1 + \frac{2B}{A} \right| \ , \tag{52}$$

and examining Equation (33) we obtain

$$\chi \sim 1 - \frac{1 - A(1-f)}{B} \ . \tag{53}$$

(In these equations we have assumed $A > 0$ so if $A < 0$ we should flip the sign of f but make sure to evaluate χ before this sign flip.) The conditions for validity of the large w expansion are:

$$A + 2B >> 0 \tag{54}$$

For values of A and B which do not satisfy the constraints of either the low w or high w limits, Equation (31) and Equation (34), and for which the real branch is operative ($B > 0$ or $w_0^2 > 0$) must be solved numerically.

To consider the imaginary branch it is convenient to define the quantities

$$z = iw \ , \quad z^2 = -w^2 \ , \quad z_0^2 = -w_0^2 \tag{55}$$

in terms of which are basic Equation (31) becomes

$$z^2 - z_0^2 = (-4B)(z \cot(z) - 1) \ . \tag{56}$$

For $|z_0^2| << 1$ the expansions of Equation (44) still apply. If $z_0^2 >> 1$ then it is clear that we are near the pole in $\cot(z)$ in Equation (56), or $z \sim \pi$. (Solutions of this transcendental equation for $z > \pi$ can be shown to violate the conditions of Equation (39).) Letting

$$z = \pi + \theta \tag{57}$$

we expand Equation (56) to quadratic order in θ which leads to

$$z \sim \pi + \theta_0 \left(1 + (1 + \frac{1}{2}((1 - 2(1 - 2B/3)/B\, \theta_0^2)^{1/2} - 1 \right)^{-1} \tag{58}$$

where

$$\theta_0 = \frac{4\pi B}{z_0^2 - \pi^2} \ . \tag{59}$$

To this order we obtain

$$\begin{aligned} f \ &\sim \ \frac{1}{4B} \left(\log \left| \frac{\pi^2 + (A-2B)^2}{\pi^2 + (A+2B)^2} \right| - 2A \right) \\ &+ \ \frac{\pi}{2B} \left(\frac{1}{\pi^2 + (A-2B)^2} - \frac{1}{\pi^2 + (A+2B)^2} \right) \theta \end{aligned}$$

$$+ \quad \frac{1}{2B}\left(\frac{1}{\pi^2+(A-2B)^2}-\frac{1}{\pi^2+(A+2B)^2}\right)$$
$$\left(1-2\pi^2(\frac{1}{\pi^2+(A-2B)^2}+\frac{1}{\pi^2+(A+2B)^2})\right)\frac{\theta^2}{2} \tag{60}$$
$$\chi \quad = \quad -\frac{1+Af}{B}-\frac{A^2+(\pi+\theta)^2}{4B^2} \ . \tag{61}$$

This expansion is valid when

$$\theta_0 << \pi \ . \tag{62}$$

4 Results

Because of the reflection property, Equation (36) we concentrate on the case $A > 0$. Figures 1 and 2 display three dimensional surface plots of f and χ as functions of A and B/A. At large values of A, corresponding to the free streaming regime, there is little change in the values of f and χ from the **LP** result that both f and χ go to unity, as long as B is positive. For negative B, however, f and χ are both significantly reduced. At small values of A, i.e. in the diffusive regime, f and χ depart significantly from the **LP** result for both positive and negative B.

Figure 3 looks in more detail at behavior of f and χ in the representative cases $A =$ 100, 10, 1, for B ranging from $-3A$ to $3A$. Both f and χ monotonically increase with B at constant A. As A becomes large both Eddington factors drop precipitously for negative B, the sharpness of the transition increasing with increasing magnitude of A. Note that for $B << 0$ it is possible that $\chi < 1/3$, the limiting factor in the case of $B = 0$. However, as long as $\chi > 0$ the results are physical.

The transition to the imaginary branch in the solution of the transcendental equations occurs when the quantity w_0^2 of Equation (40) goes negative, which occurs only for $B < 0$ when

$$B < \frac{1-(1+A^2)^{1/2}}{2} \sim -\frac{A}{2}(1-\frac{1}{A}+\frac{1}{4A^2}) \tag{63}$$

the expansion giving the case for large A. Physically we can understand the drop in f and χ by observing that the angular distribution function given by Equation (17) has an extremum at the point $\mu = -B/2A$. If $B < 0$ it is easy to verify that the function passes through a maximum at this angle. If $B < -2A$ then this maximum will occur between $\mu = 0$ and $\mu = 1$. Otherwise ψ is always monotonic, increasing as a function of μ for positive f which means the flux is moving outward, and decreasing for negative f, in which case the flux is moving inwards. As the value of A gets very large, ψ rises very rapidly at $\mu = 1$, becoming eventually a spike, as it should. However, when $B < -2A$ the angular distribution is shifted to lower μ–values; i.e., the flux doesn't move out as much and the radiation tends feels a drag from the co–moving frame. Since this occurs in an optically thin regime (low κ) this may appear to be strange because in such material one would expect the radiation to stream through essentially unperturbed. In fact, it is the drop in the Eddington factors which causes this behavior to be respected.

The case $B << 0$ tends to occur in the region just past an accretion shock, where while the velocity is negative, its gradient is large and positive. This will be seen more clearly in the following section which gives a hydrodynamic example. The problem is that we have been working in Lagrangean coordinates, which are tied to the material, yet the radiation is flowing in the optically thin regime.

5 A Hydrodynamical Example

In order to gauge the importance of correcting the **LP** flux limiter to include the velocity–dependent terms, we consider an actual hydrodynamics calculation including radiative transport. This test problem includes the propagation of strong shock waves through a medium which has both opaque and transparent regions for the radiation.

The problem considered is the propagation of a shock wave in the dense regions of an exploding supernova; it is in fact the difficulties of properly calculating radiative transport in this environment which has motivated the authors' interest in this problem, although the formalism we have presented can be applied to a wide range of problems. In the supernova the radiation of interest is neutrinos, and the calculation includes neutrinos and antineutrinos of all flavors.

The hydrodynamical code used as the testbed has been especially developed to calculate the Type II supernova mechanism, from the development of the initial instability through the birth of the neutron star. Full details will be published elsewhere (Cooperstein 1992). The hydrodynamics assumes spherical symmetry and the dynamics are fully general relativistic using the formalism as developed by Baron *et al.* 1987. The differencing scheme, both for the hydrodynamics and the radiative transport, is completely implicit, and thus is not restricted by the Courant stability condition. Shocks are represented using the pseudoviscosity method, and dynamic rezoning is employed to give fine zones near shock waves and to remove zones where they are no longer needed.

The equation of state is based on that developed by the authors (Baron, Cooperstein and Kahana 1985a,b; Cooperstein 1985; Cooperstein and Baron 1990) with some technical modifications and improvements in a number of areas such as the nuclear symmetry energy and partition function (Cooperstein 1992), and is capable of covering material both at the high density in the interior of a neutron star and the low density of a supernova envelope, at both high and low entropies, and at varying degrees of neutron richness. In regions which have not reached nuclear statistical equilibrium, nuclear burning is included in a simplified fashion.

The radiative transport method is based on the "two–fluid" model (Cooperstein, van den Horn and Baron 1986; 1987) in which the neutrinos of each type are assumed to possess a Fermi–Dirac distribution, for which the chemical potential and neutrino temperature are obtained by inverting the number and energy densities. The evolution in time of the number and energy densities and fluxes are independently followed. The neutrinos are never assumed to be in local thermal or chemical equilibrium with the material, and local source terms are evaluated for exchange of energy and lepton number with the material. Neutrino opacities are based on those given in Cooperstein, van den Horn and Baron 1986, with some updating and improvements.

A more complete discussion of a related and similar code is given in Cooperstein and Baron 1990; the present code differs mostly in the introduction of implicit hydrodynamics and improvement of details.

The initial model used is that furnished by Weaver and Woosley (1992), for a $20M_\odot$ star, evolved using restricted semi–convection and a high rate for the reaction $^{12}C(\alpha,\gamma)^{16}O$. Approximately the inner $2.5M_\odot$ of the progenitor star are followed in the calculation, with the number of Lagrangean mass shells varying from 75 to 120 according to the necessities of good dynamic resolution at various times. A "stiff" equation of state is used; the relevant parameters are a nuclear bulk incompressibility of $K_0 = 300$ MeV , high density adiabatic index $\gamma = 3$ (Cooperstein and Baron 1990; Baron and Cooperstein 1990).

The general features of the collapse and early shock wave propagation of such models has been exhaustively studied by a number of different authors and the general features are considered in Baron and Cooperstein 1990. After a homologous collapse of the inner regions, the core bounces when its density becomes several times greater than nuclear matter density at about $\rho \sim 10^{15}$ g cm^{-3} and a shock wave forms near the sonic point at $R \sim 10$ km, $M \sim 0.7M_\odot$, and then rapidly propagates outward. It proceeds out to a maximum radius which depends on the equation of state, softer equations of state producing larger excursions and then falls back, becoming an accretion shock as material continues to fall upon the collapsed core. In this model the shock reaches no more than 150 km, and then settles down at 125 km, within a matter of a few tens of milliseconds. What happens subsequently is not completely understood. The shock waves calculated by Wilson and Mayle in a number of publications (Wilson 1985; Bethe and Wilson 1985; Mayle 1990 and references therein) can be revived on a time scale of a half second or more, by the absorption of a small fraction of the neutrinos escaping the nascent neutron star at the supernova core, depending on the assumptions the authors make about how convection proceeds. While it is not the purpose of this paper to consider the viability of the so-called "delayed" mechanism, we do seek to consider some aspects of the radiative transport which are probably quite important for the mechanism's viability.

The calculation has been carried forward for a little less than 400 milliseconds after bounce of the supernova core. The accretion shock gradually moves inward as the density of the center continues to increase, until it has moved into 65 km. The neutrino–sphere, where the neutrino optical depth is of order unity, is at about 35 km for all species of neutrinos, so the general situation is of a very bright source shining though a shock wave in semi–transparent material.

The present calculation has been done by using the **LP** flux limiting scheme with the velocity dependent B–term set to zero. We calculate the change induced by calculating the proper Eddington factors with non–vanishing B. Thus, this is not a self–consistent picture. A calculation using the full scheme herein developed is under progress and results will be reported elsewhere. The special relativistic form of the flux limiter has been used within a general relativistic framework as discussed in Baron *et al.* 1987. In the regime which we consider, the general relativistic corrections are not large.

Figures 4 and 5 display the situation at $t = 0.367$ seconds after bounce. The shockwave can be seen to be at about 60 km, either by looking at the discontinuity in v, in the first part of the figure, or the factor Q/P in the second figure where Q is the pseudoviscosity and P the material pressure, which measures the strength of the shock wave. The density at the accretion shock is seen to be $\rho \sim 10^8$ g cm^{-3}. While the resolution of these calculations may appear to be not very high in fact the zones are quite small, less than $10^{-4} M_\odot$ in size, but due to the very sharp gradients involved only a few zones are in the region of the plots, and so the graphs look quite rough. This is actually partly due to the errors in the flux limiter as we shall discuss.

The remaining material in these figures refers to transport properties and we have chosen to show the results only for the electron neutrino radiation; the anti–neutrinos and pair–neutrinos show qualitatively similar behavior. Figure 4 displays the quantities A and B and the ratio B/A. We see that this regime $A \sim 500$ so we are very much in a free-streaming situation; the diffusion flux would be 500/3 times the maximum causal flux if we used unmodified diffusion theory. The factor B/A is positive interior to the shock wave, reaching a value of $B/A \sim 0.5$ and negative in the shock wave itself and past it, reaching a value $B/A \sim -0.6$ somewhat past the shock wave. The effects on the Eddington factors f and χ are also displayed in Figure 5, with the dotted lines giving the behavior for $B = 0$ and the solid lines the case for non–vanishing B. The plots have been given on two different scales in order to highlight the region between the neutrino sphere and the shock wave.

It is seen that there are two regions where the differences are significant. In the region between $R = 30$ km and $R = 40$ km, just past the neutrino sphere, the new results have larger Eddington factors. In the region ahead of the shock wave $R = 80 - 100$ km we see that there are a few zones (the resolution is not very good here) that have very large differences in the Eddington factors. In these zones the quantity B/A drops to $B/A \sim -0.6$, and as noted in the previous section it is here that the angular distribution function achieves a maximum at $\mu < 1$, and it is when B/A drops to less than -0.5 that the Eddington factors drop dramatically. The physical cause of this difference can be seen in the neutrino temperature profile across the shock wave, also displayed in Figure 4. The neutrino temperature has a minimum behind the shock wave and then rises to a maximum in the accretion region. This unphysical behavior is the result of using a poor flux limiter; the neutrinos should stream right through the shock region since the mean free path is very large. The new flux limiter is clearly trying to correct this unphysical situation by bottling up the neutrinos ahead of the shock wave, to remove the temperature gradient. We believe that this situation would not have occurred in the hydrodynamical calculation if we had used the correct flux limiter throughout and are presently carrying out such a self consistent calculation.

This unphysical radiation temperature gradient, a direct product of a bad flux limiting prescription, has unfortunate consequences for the numerical stability of a hydrodynamics calculation, as the bump in the neutrino temperature distribution can easily provoke instabilities. Furthermore, since the radiation pressure in this material is $100 - 1000$ times as great as the pressure of the matter, even a very small error in the transfer of momentum between the neutrinos and the material may have large effects and lead to incorrect results, including either the production of an explosion, or the destruction of one.

6 Gauge Freedom

Cernohorsky and van den Horn (1990) have suggested that in order to maintain the maximum entropy form of the angular dependence of the flux limiter the standard **LP** prescription should be somewhat modified. Cernohorsky and van den Horn (1990) propose that Equation (13) should be modified in the following fashion:

$$\begin{aligned} &\left\{\frac{\partial\psi}{\partial t} + \mu\frac{\partial\psi}{\partial\mu} + \frac{1-\mu^2}{R}\frac{\partial\psi}{\partial\mu} + \left(\frac{v}{R} - \frac{\partial v}{\partial R}\right)\mu(1-\mu^2)\frac{\partial\psi}{\partial\mu} - \psi\frac{1}{R^2}\frac{\partial(R^2 f)}{\partial R}\right\} \\ + \quad \psi \quad &\left\{\frac{\mu - f}{\varepsilon}\frac{\partial\varepsilon}{\partial R} + \left(\frac{v}{R} - \frac{\partial v}{\partial R}\right)(1 + \chi - 4\mu^2) + a\kappa\right\} = a\kappa. \end{aligned} \tag{64}$$

where we have moved the divergence of the flux limiter f from the second curly bracket to the first.

In this case we see that this is a poor choice, since if we demand that the first term in curly brackets go to zero and integrate over angle we find:

$$\frac{1}{R^2}\frac{\partial(R^2 f)}{\partial R} - \left(\frac{v}{R} - \frac{\partial v}{\partial R}\right)(1 - 3\chi) = \frac{1}{R^2}\frac{\partial(R^2 f)}{\partial R} \tag{65}$$

or

$$\chi = \frac{1}{3} \tag{66}$$

everywhere. Whereas this choice makes no difference in the static case since the coefficient $\left(\frac{v}{R} - \frac{\partial v}{\partial R}\right)$ is zero in that case, here the Eddington factor is forced to the diffusion limit. Thus, in the more general case it appears that the choice made by Cernohorsky and van den Horn (1990) in the static limit gives unphysical results.

7 Conclusions

We have derived a flux limiter based on the transport equation correct to order v/c. Our flux limiter reduces to that of Levermore and Pomraning (1981)(**LP**) in the case of homologous flow. We have shown that in the presence of a shock wave during supernova formation that the departures between the Eddington factors is large in the region near to the shock front. Implementing this new flux limiter into a hydrodynamic calculation from the outset has been shown to be important in the case of supernova simulations, and may be important in other physical contexts.

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Appendix

We now drop the assumption of grey opacities. We define the moments per unit frequency:

$$\varepsilon_\nu = \frac{1}{2}\int_{-1}^{+1} d\mu\, I \tag{A1}$$

$$\mathcal{F}_\nu = f_\nu\, \varepsilon_\nu = \frac{1}{2}\int_{-1}^{+1} d\mu\, \mu I_\nu \tag{A2}$$

$$P_\nu = \chi_\nu\, \varepsilon_\nu = \frac{1}{2}\int_{-1}^{+1} d\mu\, \mu^2 I_\nu . \tag{A3}$$

Taking the zeroth moment of Equation (1) we find:

$$\begin{aligned} \frac{d\varepsilon_\nu}{dt} + \frac{1}{R^2}\frac{\partial(R^2\mathcal{F}_\nu)}{\partial R} + \frac{v}{R}\left(3\varepsilon_\nu - \frac{\partial\varepsilon_\nu}{\partial\ln\nu}\right) &+ \left(\frac{v}{R} - \frac{\partial v}{\partial R}\right)\left(\frac{\partial P_\nu}{\partial\ln\nu} - \varepsilon_\nu\right) \\ &= \kappa_A(B_\nu - \varepsilon_\nu)\ . \end{aligned} \tag{A4}$$

As in Equation (10) we now introduce the energy dependent angular distribution function ψ such that

$$I_\nu \equiv \varepsilon(R,\, \nu,\, t)\psi(\mu,\, R,\, \nu,\, t) \tag{A5}$$

Using these definitions in Equation (A4) we find:

$$\begin{aligned} \frac{d\varepsilon_\nu}{dt} + \frac{1}{R^2}\frac{\partial(R^2 f_\nu\varepsilon_\nu)}{\partial R} + \frac{v}{R}\left(3\varepsilon_\nu - \frac{\partial\varepsilon_\nu}{\partial\ln\nu}\right) &+ \left(\frac{v}{R} - \frac{\partial v}{\partial R}\right)\left(\frac{\partial(\chi_\nu\varepsilon_\nu)}{\partial\ln\nu} - \varepsilon_\nu\right) \\ &= \kappa_A(B_\nu - \varepsilon_\nu) \end{aligned} \tag{A6}$$

Solving this equation for $d\varepsilon_\nu/dt$ and plugging into Equation (1), the radiation transport equation becomes:

$$\varepsilon_\nu \quad \left\{\frac{d\psi}{dt} + \mu\frac{\partial\psi}{\partial R} + (1-\mu^2)\left[\frac{1}{R} + \mu\left(\frac{v}{R} - \frac{\partial v}{\partial R}\right)\right]\frac{\partial\psi}{\partial\mu} - \left(\frac{v}{R} - \left(\frac{v}{R} - \frac{\partial v}{\partial R}\right)\mu^2\right)\frac{\partial\psi}{\partial\ln\nu}\right\}$$

$$
\begin{aligned}
+ & \quad \psi \left\{ (\mu - f_\nu) \frac{\partial \varepsilon_\nu}{\partial R} - \varepsilon_\nu \frac{1}{R^2} \frac{\partial (R^2 f_\nu)}{\partial R} + \left(\frac{v}{R} - \frac{\partial v}{\partial R} \right) \left[(1 - 3\mu^2 - \frac{\partial \chi_\nu}{\partial \ln \nu}) \varepsilon_\nu \right. \right. \\
- & \quad \left. \left. (\chi_\nu - \mu^2) \frac{\partial \varepsilon_\nu}{\partial \ln \nu} \right] + \kappa_A B_\nu + \kappa_S \varepsilon_\nu \right\} = \kappa_A B_\nu + \kappa_S \varepsilon_\nu . \qquad \text{(A7)}
\end{aligned}
$$

Once again we set the first term in curly brackets to zero and integrating over angle we find:

$$
\frac{1}{R^2} \frac{\partial (R^2 f_\nu)}{\partial R} - \left(\frac{v}{R} - \frac{\partial v}{\partial R} \right) (1 - 3\chi_\nu - \frac{\partial \chi_\nu}{\partial \ln \nu}) = 0. \qquad \text{(A8)}
$$

Solving for the divergence of f_ν and plugging into Equation (A7), we have:

$$
\left\{ (\mu - f_\nu) \frac{\partial \varepsilon_\nu}{\partial R} + \left(\frac{v}{R} - \frac{\partial v}{\partial R} \right) \left(\chi_\nu - \mu^2 \right) \left(3\varepsilon_\nu - \frac{\partial \varepsilon_\nu}{\partial \ln \nu} \right) + \kappa_\nu a_\nu \varepsilon_\nu \right\} \psi = \kappa_\nu a_\nu \varepsilon_\nu . \qquad \text{(A9)}
$$

where $\kappa_\nu = \kappa_A + \kappa_S$ is the total extinction coefficient at frequency ν and a_ν is the albedo at frequency ν such that $\kappa_\nu a_\nu \varepsilon_\nu = \kappa_A B_\nu + \kappa_S \varepsilon_\nu$. This equation may be solved for ψ to find:

$$
\psi = \frac{1}{1 + A(f_\nu - \mu) + B \frac{(3+C)}{4} (\chi_\nu - \mu^2)}, \qquad \text{(A10)}
$$

where

$$
\begin{aligned}
A &= -\frac{1}{\kappa_\nu a_\nu \varepsilon_\nu} \frac{\partial \varepsilon_\nu}{\partial r} && \text{(A11)} \\
B &= \frac{4}{\kappa_\nu a_\nu} \left(\frac{v}{R} - \frac{\partial v}{\partial R} \right) && \text{(A12)} \\
C &= -\frac{\partial \ln \varepsilon_\nu}{\partial \ln \nu}. && \text{(A13)}
\end{aligned}
$$

These equations are identical to the ones obtained in the grey case if we make the identification

$$
B \longrightarrow \frac{B(3 + C)}{4} \qquad \text{(A14)}
$$

and use the frequency dependent quantities in place of the grey ones, and hence may be solved in an identical manner.

In the case where the radiation follows a local thermodynamic distribution

$$
\varepsilon_\nu = \frac{\nu^3}{\exp((\nu - \mu)/T) \pm 1} \qquad \text{(A15)}
$$

where μ is the radiation chemical potential and T is the radiation temperature, the plus sign maintaining for fermions such as neutrinos and the minus sign for bosons such as photons (which have $\mu = 0$), we find

$$
C + 3 = \frac{\nu/T}{1 \pm \exp((\mu - \nu)/T)} \ . \qquad \text{(A16)}
$$

In the case of a degenerate Fermi system, $\mu >> 0$ we find that the velocity–dependent correction, the B–term, becomes vanishingly small for the low frequency radiation. In any case, at high frequency we find $C + 3 \sim \nu/T$ so the correction becomes most important for the high frequency part of the spectrum.

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Figure Captions

Figure 1: The Eddington factor f, giving the ratio of the energy flux to the energy density, is displayed as a three dimensional surface over the A and B/A plane. The largest differences from the $B = 0$ value are seen for negative B.

Figure 2: The Eddington factor χ, giving the ratio of the radiation pressure to the energy density, is displayed as a three dimensional surface over the A and B/A plane. The largest

differences from the $B = 0$ value are seen for negative B.

Figure 3: The Eddington factors f and χ are displayed for the cases $A = 100$, 10, 1 for B ranging from $-3A$ to $3A$. There is a large reduction in the value of f and χ at negative B.

Figure 4: Hydrodynamical results at t = 0.367 s after bounce of a supernova. The abscissa is the radius R, in units of 10 km. Displayed are: v, the material velocity in units of 10^9 cm s^{-1}; the logarithm of the density ρ, in cgs units; Q/P, the ratio of the pseudoviscosity to the material pressure, a measure of the strength of the shock; the parameters of the flux limiter, A and B/A, and T_ν, the neutrino temperature, in MeV.

Figure 5: Hydrodynamical results at t = 0.367 s after bounce of a supernova. The abscissa is the radius R, in units of 10 km, shown on two different scales, the lower scale highlighting the region between the neutrinosphere and the shock. Displayed are the Eddington factors f and χ. The dashed lines give results for $B = 0$, the solid lines for the actual B value from Figure 4.